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# Universality of level correlation function of sparse random matrices

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**Abstract.** The statistical properties of sparse random matrices ensembles are investigated by means of a supersymmetric approach with the use of a functional generalization of the Hubbard-Stratonovich (HS) transformation. When used to calculate the density of states the method is shown to be absolutely equivalent to the replica trick. The model turns out to bear a close resemblance to the Anderson model on the Bethe lattice: it possesses a delocalization transition that occurs with an increase in the 'mean connectivity' parameter. In the delocalized phase the level-level correlation function proves to have a universal (Dyson) form with the full density of states replaced by the contribution from the infinite cluster.

## 1. Introduction

The problem of level correlations of large random matrices (RM) has been investigated intensively since the early 1950s [1-6]. There has been a considerable growth of interest in this problem in the last decade due to the wide range of applications of RM theory to different branches of physics [7-9], the theory of mesoscopic fluctuations in disordered conductors [10] and the quantum mechanical aspects of chaos [11] can be mentioned as examples in this respect.

The case of independent, identically distributed matrix elements is the simplest to investigate. Assuming the distribution to be Gaussian three different RM ensembles introduced by Wigner and Dyson are possible: the orthogonal ensemble (symmetric matrices), the unitary ensemble (Hermitian matrices) and the symplectic ensemble [4-6]. Expressions for the eigenvalue correlation functions for these ensembles were presented in [2-4]. Dyson put forward the hypothesis that the form of the pair correlation function of RM eigenvalues is universal, i.e. it does not depend on the probability distribution of the RM elements and is determined by matrix symmetry only. By using reasonable assumptions of different kinds this universality has been proved in a number of papers [4, 12].

Efetov [13] suggested a method for investigating RM properties based on supersymmetry which would enable the expressions for the eigenvalue correlators for the previously mentioned ensembles to be derived more rigorously [13, 14]. Fairly recently a new kind of ensemble (that of sparse RM) attracted the attention of physicists [15, 16], due to its close connections with some spin-glass models [17], and combinatoric optimization problems [18]. These $N\times N$ $(N\to\infty)$ real symmetric matrices have a

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finite mean number $p$ of non-zero elements per row. The density of states (DOS) for such an ensemble was investigated in [15] by means of the replica trick, and an integral equation was obtained giving the basic possibility of extracting the DOS. An iteration of this integral equation resulted in a perturbative expansion for the DOS in powers of $1/p$, with the leading term reproducing the DOS of the Gaussian orthogonal ensemble (GOE) (Wigner semicircular law). It is natural to study the question of the applicability of the Dyson universality hypothesis to the ensemble of sparse RM. This is the main subject of the present paper.

As demonstrated in section 2, the ensemble of sparse RM is the only non-trivial ensemble with independent identically distributed matrix elements that differs from the Gaussian one. Besides, one of its interesting properties is the disintegration of a random matrix into disconnected blocks of finite (i.e. much less than $N$) dimension (finite clusters) when $p<p_c=1$. Clearly, this should result in the absence of eigenvalue correlations at distances of the order of $1/N$. When $p>p_c$ a connected block of dimension $D\sim N$ appears (infinite cluster). It is far from being clear whether the Dyson universality [4, 12] is maintained under these conditions. All this gives a special interest to the problem under investigation. Let us remind the reader that in accordance with [14] the eigenvalue correlator could not be calculated correctly be means of the replica trick. So, we use a supersymmetric approach to the problem following the ideas in [13, 14].

The outline of the paper is as follows. In section 2 within the scope of supersymmetric approach we rederive an integral equation which gives rise to the possibility of extracting the DOS which was previously derived in [15] by the replica trick. A power expansion in $p\ll 1$ reproduces a contribution of finite clusters into the DOS. Section 3 is devoted directly to the calculation of the level correlator. The similarlity between the Anderson model on the Bethe lattice and the model under investigation is revealed and it is shown that the level-level correlation function has the Dyson form within the delocalized phase with the full DOS being replaced by the infinite cluster contribution to it.

When we had completed this work we learnt about a paper by Rodgers and De Dominicis [19], where a supersymmetric method was applied to the calculation of the DOS of sparse RM. However, the authors of [19] were able to show the equivalence of the supersymmetric and replica approaches only when $p\to\infty$. A comment on this issue is the subject of a separate publication [20]. The problem of calculating the level correlation function was not addressed in [19].

## 2. The density of states

We are going to study the statistical properties of a real, symmetric $N\times N$ matrix $\hat{H}$ $(N\to\infty)$ whose elements $H_{ij}$ $(=H_{ji})$ are independent, identically distributed random variables, with a certain even probability distribution function $f(H_{ij})$. As usual, it is convenient to require the characteristic magnitude of the eigenvalues $\lambda_i$ to be of the order of unity. Then $\Sigma_{ij}H_{ij}^2=\mathrm{Tr}\,H^2=\Sigma_i\lambda_i^2\sim N$ and therefore $\langle H_{ij}^2\rangle\sim N^{-1}$, where angular brackets denote an averaging over the distribution function $f$. A general form of $f(H_{ij})$ satisfying that condition is

$$f(z)=(1-a)\delta(z)+ah(z)\qquad 0\leqslant a\leqslant 1 \tag{1}$$

where it is assumed that $h(z)$ has no $\delta$-like singularity at $z=0$ and $a\int h(z)z^2\,\mathrm{d}z\sim N^{-1}$.

Hence

$$a \sim N^{-\alpha} \qquad B = \int h(z) z^2 \, \mathrm{d}z \sim N^{\alpha-1} \qquad 0 \leqslant \alpha \leqslant 1. \tag{2}$$

The DOS of a random matrix $H_{ij}$ within the scope of the supersymmetric approach is given by the following expression [14, 21].

$$\rho(E) = (2\pi N)^{-1} \operatorname{Im} \frac{\partial}{\partial J} \langle Z(E, J) \rangle |_{J=0}$$

$$Z(E, J) = \int \prod_i [\mathrm{d}\phi_i] \exp\left\{ \frac{i}{2} \sum_{ij} \phi_i^+ [(E\hat{I} + J\hat{K})\delta_{ij} - H_{ij}] \phi_j \right\} \tag{3}$$

where

$$\phi_i = \begin{pmatrix} S_i^{(1)} \\ S_i^{(2)} \\ \chi_i \\ \chi_i^* \end{pmatrix} \qquad \phi_i^+ = (S_i^{(1)}, S_i^{(2)}, \chi_i^*, -\chi_i)$$

is a supervector with two real commutative components $S_i^{(1)}$, $S_i^{(2)}$ and two Grassmannian components $\chi_i, \chi_i^*$; $[\mathrm{d}\phi_i] = \mathrm{d}S_i^{(1)} \, \mathrm{d}S_i^{(2)} \, \mathrm{d}\chi_i^* \, \mathrm{d}\chi_i$

$$\hat{K} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

and $\hat{I}$ is the identity matrix†. By averaging we get

$$\langle Z(E, J) \rangle = \int \prod_i [\mathrm{d}\phi_i] \exp\left\{ \frac{1}{2} \sum_i \phi_i^+ (E\hat{I} + J\hat{K}) \phi_i + \sum_{i,j} \ln\left\langle \exp\left( -\frac{\mathrm{i}}{2} H_{ij} \phi_i^+ \phi_j \right) \right\rangle \right\}. \tag{4}$$

Then condition (2) can be represented in the following form

$$h(z) = h_1(zN^{(1-\alpha)/2}) N^{(1-\alpha)/2} \tag{5}$$

where $h_1(z) \sim 1$ when $z \sim 1$. Then

$$\ln\left\langle \exp\left( -\frac{\mathrm{i}}{2} H_{ij} \phi_i^+ \phi_j \right) \right\rangle = \ln\left\{ 1 + a\left[ \int \exp\left( -\frac{\mathrm{i}}{2} N^{(\alpha-1)/2} z \phi_i^+ \phi_j \right) h_1(z) \, \mathrm{d}z - 1 \right] \right\}. \tag{6}$$

Depending on the value of $\alpha$, two essentially different cases in equation (6) are possible. For $\alpha < 1$ and $N \to \infty$ we can truncate the series expansion to the first non-vanishing term. So we have

$$\ln\left\langle \exp\left( -\frac{i}{2} H_{ij} \phi_i^+ \phi_j \right) \right\rangle = -\tfrac{1}{8} a N^{\alpha-1} (\phi_i^+ \phi_j)^2 \int z^2 h_1(z) \, \mathrm{d}z. \tag{7}$$

Therefore, in case $\alpha < 1$ any ensemble (1) is equivalent to the Gaussian one with the distribution

$$f(z) = \sqrt{\frac{N}{2\pi\sigma}} \exp\left\{ -N \frac{z^2}{2\sigma} \right\}$$

where $\sigma = aN^{\alpha} \int h_1(z) z^2 \, \mathrm{d}z = Na \int h(z) z^2 \, \mathrm{d}z$.

† To find the definitions of supervectors and simplest rules of handling them see e.g. [13, 21, 22].

Thus, the only non-trivial (i.e. irreducible to Gaussian) case is $\alpha=1$. Setting $a=p/N$, $p\sim 1$, $N\to\infty$ we get from equations (4)-(6):

$$\langle Z(E,J)\rangle = \int \prod_i [d\phi_i] \exp\left\{\frac{\mathrm{i}}{2}\sum_i \phi_i^+(E\hat{I}+J\hat{K})\phi_i + \frac{p}{2N}\sum_{ij}[\tilde{h}(\phi_i^+\phi_j)-1]\right\} \tag{8}$$

$$\tilde{h}(z)=\int h(t)\exp(-\mathrm{i}tz)\,\mathrm{d}t.$$

To proceed further we have to decouple the variables $\phi_i$ connected with different sites $i$. The authors of papers [15, 19], following the method developed in [23], expanded function $\tilde{h}$ as a power series and decoupled every term by the introduction of auxiliary variables (Hubbard–Stratonovich (HS) transformation) in the usual way. The integration over the auxiliary variables could be performed for $N\to\infty$ by the steepest descent method resulting in an infinite set of coupled saddle-point equations. Introducing a generating function one succeeds in rewriting this set in terms of a single integral equation.

Instead of the previously mentioned procedure we suggest using a functional generalization of the HS transformation:

$$\begin{aligned}\int \mathrm{D}g\exp\left\{-\frac{Np}{2}\int[\mathrm{d}\psi][\mathrm{d}\psi']g(\psi)C(\psi,\psi')g(\psi')+p\int \mathrm{d}[\psi]g(\psi)v(\psi)\right\}\\ =\exp\left\{\frac{p}{2N}\int[\mathrm{d}\psi][\mathrm{d}\psi']v(\psi)C^{-1}(\psi,\psi')v(\psi')\right\}\end{aligned} \tag{9}$$

where $C^{-1}(\psi,\psi')$ denotes the kernel of an integral operator inverse to those with kernel $C(\psi,\psi')$.

Inserting in this identity $v(\psi)=\Sigma_{i=1}^N\delta(\psi-\phi_i)$ and choosing kernel $C^{-1}(\theta,\phi)$ to be equal to a function $\tilde{h}(\theta^+\phi)-1$, we come to the following expression:

$$\begin{aligned}\exp\left\{\frac{p}{2N}\sum_{ij}[\tilde{h}(\phi_i^+\phi_j)-1]\right\}\\ =\int \mathrm{D}g\exp\left\{-\frac{Np}{2}\int[\mathrm{d}\psi][\mathrm{d}\psi']g(\psi)C(\psi,\psi')g(\psi')+p\sum_i g(\phi_i)\right\}\end{aligned} \tag{10}$$

where $C(\psi,\psi')$ is determined by the relation

$$\int[\mathrm{d}\chi]C(\phi,\chi)[\tilde{h}(\chi^+\eta)-1]=\delta(\phi,\eta) \tag{11}$$

$\delta(\phi,\eta)$ being $\delta$-function in the space of supervectors.

It can be proved that the integral operator with the kernel $\tilde{h}(\phi^+\chi)-1$ can be inverted in the space of even functions $g(\phi)$. We discuss this matter in more detail at the end of this section.

Using (10) we transform (8) to the form:

$$\begin{aligned}\langle Z(E,j)\rangle = \int \mathrm{D}g\exp\left\{-\frac{Np}{2}\int[\mathrm{d}\psi][\mathrm{d}\psi']g(\psi)C(\psi,\psi')g(\psi')+N\ln\int[d\phi]\right.\\ \left.\times\exp\left[\frac{\mathrm{i}}{2}\phi^+(E\hat{I}+J\hat{K})\phi+pg(\phi)\right]\right\}.\end{aligned} \tag{12}$$

Performing the functional integration over $g$ for $N\to\infty$ by the steepest descent method, we get (for $J=0$) the following saddle-point equation for the function $g(\phi)$:

$$g(\psi)=\frac{\int[\mathrm{d}\phi]\{\hat{h}(\phi^{+}\psi)-1\}\exp((\mathrm{i}/2)\phi^{+}E\phi+pg(\phi))}{\int[\mathrm{d}\phi]\exp((\mathrm{i}/2)\phi^{+}E\phi+pg(\phi))}. \tag{13}$$

In view of the invariance of equation (13) with respect to the transformation $g(\phi)\to g(\hat{T}\phi)$, $\hat{T}$ being an arbitrary unitary supermatrix, it is natural to search for its solution as a function $\tilde{g}$ of the invariant $\phi^{+}\phi=S^2+2\chi^{*}\chi$; $S^2=(S^{(1)})^2+(S^{(2)})^2$. In this case the denominator in equation (13) turns out to be unity (the general problem of an integration of invariant functions over supervectors was considered in detail in [14, 24]). Since $\tilde{g}(\phi^{+}\phi)=\tilde{g}(S^2)+2\chi^{*}\chi\tilde{g}'(S^2)$ equation (13) after the integration over Grassmannian components of the supervector $\phi$ takes the form

$$\tilde{g}(S^2)=-S\int_0^{\infty}\mathrm{d}R\exp\left[\frac{\mathrm{i}}{2}ER^2+p\tilde{g}(R^2)\right]\int\mathrm{d}z\,zh(z)J_1(zRS). \tag{14}$$

The DOS is related to the function $\tilde{g}(S^2)$ as follows

$$\rho(E)=-\frac{2}{\pi B}\operatorname{Re}\tilde{g}'(0)\qquad B=\int\mathrm{d}z\,h(z)z^2. \tag{15}$$

Here and afterwards in this section a prime denotes the derivative of a function over its argument.

Equation (14) was obtained in [15] by the replica trick with an additional assumption that the solution to the problem is replica symmetric. In some way a supersymmetric approach could be considered as a specific variant of the replica approach with half the replicated fields being anticommutative variables. From this point of view our assumption that $g$ is a function of $\phi^{+}\phi$ only seems to be equivalent to a replica symmetric ansatz. Unfortunately, we are unable to prove the absence of solutions without this symmetry, but we believe, that it is the only 'symmetric' solution that is relevant for the problem under consideration. The authors of [19] sought a solution of equation (13) in a more general form

$$g(\phi)=A(S^2)+2\chi^{*}\chi B(S^2) \tag{16}$$

that results in three coupled equations for the functions $A$, $B$ and $\tilde{Z}$, where $\tilde{Z}$ denotes the denominator on the right-hand side of equation (13). One can easily make sure that the condition $A'=B$ is consistent with that system of equations and reduces it to a single equation equivalent to equation (14). Therefore, we have shown that the supersymmetric approach is absolutely equivalent to the replica trick for the problem of calculation of the density of states.

Let us now discuss the question of the existence of the quantity $C$, defined as the kernel of the operator inverse to $\tilde{h}-1$ (see equation (11)). Such an inversion could be performed only in the absence of zero eigenvalues of the operator $\tilde{h}-1$. According to the definition, eigenfunctions $f$ and eigenvalues $\lambda$ of that operator satisfy the following equation

$$\int[\mathrm{d}\psi]\{\tilde{h}(\phi^{+}\psi)-1\}f(\psi)=\lambda f(\phi). \tag{17}$$

Let us look for eigenfunctions in a general form

$$f(\phi)=f_1(S^{(1)},S^{(2)})+f_2(S^{(1)},S^{(2)})\chi^{*}\chi. \tag{18}$$

Performing the integration over Grassmanian variables $\chi^*, \chi$ we reduce equation (17) to the system of integral equations:

$$-\int \frac{\mathrm{d}R^{(1)}\,\mathrm{d}R^{(2)}}{2\pi}[\tilde{h}(\boldsymbol{RS})-1]f_2(\boldsymbol{R}) = \lambda f_1(\boldsymbol{S})$$
$$-\int \frac{\mathrm{d}R^{(1)}\,\mathrm{d}R^{(2)}}{2\pi}\tilde{h}''(\boldsymbol{RS})f_1(\boldsymbol{R}) = \lambda f_2(\boldsymbol{S}). \tag{19}$$

Taking into account the invariance of this system with respect to O(2)-rotations we should look for its solution in the form:

$$f_t(\boldsymbol{S}) = f_t^{(m)}(S^2)\exp(\mathrm{i}m\phi_s) \qquad t=1,2$$

where $\phi_s$ is the polar angle of the vector $\boldsymbol{S}$; and $m$ is an integer number.

Performing the integration over angular variables we get

$$-\mathrm{i}^m\int \mathrm{d}R\,R\int \mathrm{d}z\,h(z)[J_m(zRS)-\delta_{0m}]f_2^{(m)}(R^2) = \lambda f_1^{(m)}(S^2)$$
$$-\mathrm{i}^m\int \mathrm{d}R\,R\int \mathrm{d}z\,h(z)J_m(zRS)f_1^{(m)}(R^2)) = \lambda f_2^{(m)}(S^2). \tag{20}$$

Due to the fact that $h(z)=h(-z)$ the left-hand side of equations (21) vanishes for odd $m$ and any $f_{1,2}^{(m)}(S^2)$, so $\lambda=0$. That means, that we should consider the operator $\tilde{h}-1$ acting within the space of even functions $f(\boldsymbol{S})=f(-\boldsymbol{S})$.

For even $m\neq 0$ and special form of distribution function $h(z)=\frac{1}{2}[\delta(z-1)+\delta(z+1)]$ investigated in [15, 19], an integral transform in the left-hand side of equation (20) is nothing but the well known Hankel transform and we immediately find that $\lambda=\pm 1$. It is easy to show that $\lambda=\pm 1$ for $m=0$ as well. It is possible to prove the absence of zero modes for more general distribution functions $h(z)$ also, so that an operator inverse to $\tilde{h}-1$ exists in the space of even functions for any reasonable distribution function $h(z)$.

As previously mentioned, equation (14) was investigated in [15] in the limit of $p\gg 1$, the leading term of an expansion in a power series of $1/p$ reproduces the Wigner 'semicircular' law for $\rho(E)$. The tails of the DOS are described by an expression which is non-perturbatiove with respect to $1/p$ ($\sim\exp(-p)$).

It seems useful to us to study equation (14) for small $p\ll 1$ as well. Let us write its solution as a power series

$$g(S^2) = g^{(0)}(S^2) + pg^{(1)}(S^2) + p^2 g^{(2)}(S^2) + \dots \tag{21}$$

Solving equations (14) by iteration we get

$$g^{(0)}(S^2) = -\int \mathrm{d}z\,h(z)\left[1-\exp\left(\frac{-\mathrm{i}z^2S^2}{2E}\right)\right]$$
$$g^{(1)}(S^2) = \int \mathrm{d}y\,h(y)\int \mathrm{d}z\,h(z)\left\{\exp\left[\frac{-\mathrm{i}z^2S^2}{[2(E-(y^2/E))]}\right]-\exp\left(\frac{-\mathrm{i}z^2S^2}{2E}\right)\right\} \tag{22}$$

and so on. For the DOS we have correspondingly

$$\rho(E) = \rho^{(0)}(E) + p\rho^{(1)}(E) + p^2\rho^{(2)}(E) + \dots \tag{23}$$

$$\rho^{(0)}(E) = \delta(E) \qquad \rho^{(1)}(E) = h(E) - \delta(E), \dots . \tag{24}$$

This expansion is quite transparent and corresponds to step-by-step consideration of contributions of finite clusters of growing range. The simplest cluster corresponding to a row with all elements equal to zero gives the contribution to $\rho(E)$ of the form $\exp(-p)\delta(E)=(1-p+\ldots)\delta(E)$. The following cluster is generated when two sites break away: $J_{mn}\neq 0$; $J_{mi}=0$ for any $i\neq n$ and $J_{nj}=0$ for any $j\neq m$. Its contribution to $\rho(E)$ has the form $\sim p\exp(-2p)h(E)$. Evidently, these contributions are reproduced correctly by (24). We should remark that the infinite cluster arises only above the percolation threshold $p_c=1$ and gives no contribution to the series expansion in powers of $p$. Meanwhile, contributions of finite clusters are not taken into account by the series expansion in powers of $1/p$ of [15], as they have a non-perturbative form with respect to $1/p$ (they are proportional to $e^{-mp}$).

## 3. Level correlation function

Let us determine a correlator of eigenvalues in a standard way

$$\begin{aligned} S(E,r)&=\left\langle\rho\left(E+\frac{r}{2N}\right)\rho\left(E-\frac{r}{2N}\right)\right\rangle-\rho^2(E)\\ &=\frac{1}{2\pi^2}\operatorname{Re}K_{\text{con}}(E,r) \end{aligned}\tag{25}$$

where

$$\begin{aligned} &K_{\text{con}}(E,r)=K(E,r)-N^{-2}\langle\operatorname{Tr}(E_1-\hat{H})^{-1}\rangle\langle\operatorname{Tr}(E_2-\hat{H})^{-1}\rangle\\ &K(E,r)=N^{-2}\langle\operatorname{Tr}(E_1-\hat{H})^{-1}\operatorname{Tr}(E_2-\hat{H})^{-1}\rangle\\ &E_1=E+\frac{r}{2N}+\mathrm{i}\varepsilon\qquad E_2=E-\frac{r}{2N}-\mathrm{i}\varepsilon\qquad \varepsilon\to+0. \end{aligned}\tag{26}$$

Introducing a double set of variables $\phi_i^{(s)}$, $s=1,2$, we can write $K(E,r)$ in the form

$$K(E,r)=(2N)^{-2}\frac{\partial^2}{\partial J^{(1)}\,\partial J^{(2)}}\langle Z(E,r,J^{(s)})\rangle|_{J^{(s)}=0}\tag{27}$$

$$Z(E,r,J^{(s)})=\int\prod_{i,s}[\mathrm{d}\phi_i^{(s)}]\exp\left\{\frac{\mathrm{i}}{2}\sum_{i,j,s}\phi_i^{(s)+}D_{ij}^{(s)}L_s\phi_j^{(s)}\right\}\tag{28}$$

$$D_{ij}^{(s)}=\left[E+\left(\frac{r}{2N}+\mathrm{i}\varepsilon\right)L_s+J^{(s)}\hat{K}\right]\delta_{ij}-H_{ij}\qquad L_s=(-1)^{s-1}.$$

For the sake of convenience we will omit the symbol of the identity matrix $I$ henceforth.

Averaging equation (28) we get

$$\begin{aligned} \langle Z(E,r,J)\rangle=\int\prod_i[\mathrm{d}\phi_i]\exp\bigg\{&\frac{\mathrm{i}}{2}\phi^+\hat{L}(E+\hat{J}\hat{K})\phi+\frac{\mathrm{i}}{2}\phi^+\left(\frac{r}{2N}+\mathrm{i}\varepsilon\right)\phi\\ &+\frac{p}{N}\sum_{i<j}[\tilde{h}(\phi_i^+\hat{L}\phi_j)-1]\bigg\} \end{aligned}\tag{29}$$

where we have united supervectors $\phi_i^{(s)}$, $s=1,2$ into a single eight-component supervector

$$\phi_i=\begin{pmatrix}\phi_i^{(1)}\\ \phi_i^{(2)}\end{pmatrix}$$

and $\hat{L}, \hat{J}$ are now diagonal $8\times 8$ matrices with elements $L_s$ and $J^{(s)}$ correspondingly. Now we can use the functional generalization of the HS transformation in (10) again, but with $\tilde{h}(\phi_i^+\hat{L}\phi_j)$ substituting for $\tilde{h}(\phi_i^+\phi_j)$. As a result we obtain:

$$\langle Z(E,r,J)\rangle = \int \mathrm{D}g \exp\left\{-\frac{Np}{2}\int [\mathrm{d}\psi][\mathrm{d}\psi']g(\psi)C(\psi,\psi')g(\psi') + N\ln\int [\mathrm{d}\phi] \times \exp\left[\frac{\mathrm{i}}{2}\phi^+\left[\hat{L}(E+\hat{J}\hat{K}) + \frac{r}{2N} + \mathrm{i}\varepsilon\right]\phi + pg(\phi)\right]\right\}. \tag{30}$$

Then the saddle-point equation has the following form:

$$g(\psi) = \frac{\int [\mathrm{d}\phi][\tilde{h}(\phi^+\hat{L}\psi) - 1]\exp[(\mathrm{i}/2)E\phi^+\hat{L}\phi - (\varepsilon/2)\phi^+\phi + pg(\phi)]}{\int [\mathrm{d}\phi]\exp[(\mathrm{i}/2)E\phi^+\hat{L}\phi - (\varepsilon/2)\phi^+\phi + pg(\phi)]}. \tag{31}$$

The structure of equation (31) suggests an idea to seek its solution in the form of a function of two invariants: $g_0(x,y)$; $x=\phi^+\phi$, $y=\phi^+\hat{L}\phi$. Then the denominator in equation (31) turns out to be equal to unity for the same reason as in equation (13).

It is easy to notice that equation (31) is invariant (for $\varepsilon\to 0$) with respect to a transformation $g(\phi)\to g(\hat{T}\phi)$, a supermatrix $\hat{T}$ satisfying the condition

$$\hat{T}^+\hat{L}\hat{T} = \hat{L}. \tag{32}$$

Thus, if $g_0(\phi^+\phi, \phi^+\hat{L}\phi)$ is a saddle-point, then equation (31) is satisfied as well by any function $g_T(\phi)$ of the following form:

$$g_T(\phi) = g_0(\phi^+\hat{T}^+\hat{T}\phi, \phi^+\hat{L}\phi). \tag{33}$$

Therefore there is a whole family of saddle-point functions $g_T(\phi)$ giving a contribution to the integral (30) calculated by the steepest descent method. To separate the integration over that manifold in an explicit way, it is convenient to make a shift in a variable $g(\phi)$ under sign of integral (30):

$$g(\phi)\to g(\phi) + \frac{\mathrm{i}}{2p}\phi^+\left(\hat{L}\hat{J}\hat{K} + \frac{r}{2N}\right)\phi. \tag{34}$$

Then equation (30) reduces to the form

$$\langle Z(E,r,J)\rangle = \int \mathrm{D}\mu(\hat{T})\exp\left\{\frac{\mathrm{i}N}{2}\int [\mathrm{d}\psi][\mathrm{d}\psi']g_T(\psi)C(\psi,\psi')\psi'^+\left(\hat{L}\hat{J}\hat{K} + \frac{r}{2N}\right)\psi'\right\}. \tag{35}$$

According to the definition of the kernel $C(\psi,\psi')$(cf (11)) we have

$$\int [\mathrm{d}\psi][\mathrm{d}\psi']C(\psi,\psi')g(\psi)[\tilde{h}(\psi'^+\hat{L}\eta) - 1] = g(\eta). \tag{36}$$

Expanding equation (36) in a power series of $\eta$ to terms of the order of $\eta^2$, we get

$$\int [\mathrm{d}\psi][\mathrm{d}\psi']g(\psi)C(\psi,\psi')\psi'_\alpha\psi'^+_\beta = -\frac{1}{B}(\hat{L}\hat{G}\hat{L})_{\alpha,\beta} \tag{37}$$

$$G_{\alpha,\beta} = \frac{\vec{\partial}^2 g(\eta)}{\vec{\partial}\eta^+_\alpha\,\overleftarrow{\partial}\eta_\beta}\bigg|_{\eta=0} \tag{38}$$

where indices $\alpha,\beta = 1,\ldots,8$ in equations (37) and (38) number components of supervectors. It goes without saying that the difference between left and right derivatives in equation (38) make sense only for the differentiation with respect to Grassmannian components of a supervector.

Inserting equation (33) into (38) we have

$$G^T_{\alpha,\beta} \equiv \frac{\partial^2 g_T(\eta)}{\partial \eta^+_\alpha \partial \eta_\beta}\bigg|_{\eta=0} = 2g_{0x}(\hat{T}^+\hat{T})_{\alpha,\beta} + 2g_{0y}L_{\alpha,\beta} \tag{39}$$

where

$$g_{0x} = \frac{\partial g_0(x,y)}{\partial x}\bigg|_{x,y=0} \qquad g_{0y} = \frac{\partial g_0(x,y)}{\partial y}\bigg|_{x,y=0}.$$

Using equations (37) and (39) we transform (35) into the following form:

$$\langle Z(E,r,J)\rangle = \int \mathrm{D}\mu(\hat{T}) \exp\left\{-\frac{\mathrm{i}N}{B}\mathrm{Str}[g_{0x}\hat{L}\hat{T}^+\hat{T}\hat{L} + g_{0y}\hat{L}]\left[\hat{L}\hat{J}\hat{K} + \frac{r}{2N}\right]\right\} \tag{40}$$

where symbol Str stands for the supertrace [13, 21, 22].

Now it is necessary to dwell on the question about a region and a measure of integration in (35) and (40). Applied to Gaussian ensembles this was originally considered by Efetov [13] and it was then carefully worked out in [14, 21, 24]. The analysis carried out there turns out to be applicable to our case and shows that a region of integration is the graded coset space UOSP(2, 2/2, 2)/UOSP(2, 2) × UOSP(2, 2) and $\mathrm{D}\mu(T)$ is the corresponding invariant measure.

Before obtaining the final expression for $K(E, r)$ we find the expression for the DOS in terms of the functions $g_0(x, y)$. We have (cf (3) and (15))

$$\begin{aligned}\rho(E) &= \frac{1}{4\pi\mathrm{i}N}\sum_p L_p \frac{\partial}{\partial J^{(p)}}\langle Z(E,r,J)\rangle|_{J(p)=0}\\ &= -\frac{g_{0x}}{4\pi B}\int \mathrm{D}\mu(\hat{T})\,\mathrm{Str}(\hat{K}\hat{T}^+\hat{T})\exp\left\{-\frac{\mathrm{i}r}{2B}g_{0x}\,\mathrm{Str}(\hat{T}^+\hat{T})\right\}.\end{aligned} \tag{41}$$

The general method of calculation of integrals over graded coset space was presented in [14, 21, 24]. For the present case it reduces to the fact that the integral in (41) is equal to the integrand with the matrix $\hat{T}$ replaced by the identity matrix. Finally we obtain

$$\rho(E) = -\frac{2}{\pi B}g_{0x}. \tag{42}$$

This expression has a quite transparent meaning. Indeed, so long as $\rho(E)$ does not change on a scale of the order of $N^{-1}$, we can let $r$ tend to infinity keeping $r \ll N$ during the calculation. Then it is the single saddle-point function $g_0(x, y)$ that contributes to the integral (30) for $\langle Z(E, r, J)\rangle$. Calculating this integral by the steepest descent method we come back to the expression (42).

Returning to the computation of the correlator $K(E, r)$ we have from (27) and (40):

$$\begin{aligned}K(E,r) = \int \mathrm{D}\mu(\hat{T})\frac{1}{4B^2}&\mathrm{Str}\left\{[g_{0x}\hat{T}^+\hat{T} + g_{0y}]\frac{1+\hat{L}}{2}\hat{K}\right\}\\ &\times \mathrm{Str}\left\{[g_{0x}\hat{T}^+\hat{T} - g_{0y}]\frac{1-\hat{L}}{2}\hat{K}\right\}\exp\left(-\frac{\mathrm{i}r}{2B}g_{0x}\,\mathrm{Str}\,\hat{T}^+\hat{T}\right).\end{aligned} \tag{43}$$

Subtracting a disconnected part of the correlation function

$$K_{\mathrm{disc}}(E, r) \equiv K(E, r \to \infty) = \frac{4}{B^2}(g_{0x}^2 - g_{0y}^2)$$

we get

$$K_{\mathrm{con}}(E, r) = \frac{4g_{0x}^2}{B^2}\left[\int \mathrm{D}\mu(\hat{T})\, \mathrm{Str}\left[\hat{T}^{+}\hat{T}\frac{1+\hat{L}}{2}\frac{\hat{K}}{4}\right] \mathrm{Str}\left[\hat{T}^{+}\hat{T}\frac{1-\hat{L}}{2}\frac{\hat{K}}{4}\right]\right.$$
$$\left. \times \exp\left\{-\frac{\mathrm{i}r}{2B} g_{0x}\, \mathrm{Str}\, \hat{T}^{+}\hat{T}\right\} - 1\right]. \tag{44}$$

This result coincides exactly with the corresponding expression for the Gaussian orthogonal ensemble obtained by means of a supersymmetric approach [13, 14, 21] with a natural replacement of the density of states of GOE by $\rho(E)$ from (42). A calculation of the integral [13] leads to Dyson expression [5]:

$$\frac{1}{\rho^2(E)} S_{\mathrm{GOE}}(E, r) = 1 - \frac{\sin^2 r_e}{r_e^2} - \frac{\mathrm{d}}{\mathrm{d}r_e}\left(\frac{\sin r_e}{r_e}\right)\int_1^\infty \frac{\sin r_e t}{t}\,\mathrm{d}t \qquad r_e = \pi r \rho(E).$$

So, the Dyson universality hypothesis seems to be proved for the case of sparse random matrices. However, the following contradiction arises at this point. Indeed, as previously mentioned, the infinite cluster is absent when $p < p_c = 1$. Therefore, the correlator $S_{\mathrm{con}}(E, r)$ has to vanish. On the other hand, as the DOS is finite, (42) and (44) give a non-vanishing value of $S_{\mathrm{con}}(E, r)$. To analyse this question let us investigate the analytical properties of the solution of (31) when $p \ll 1$. Using an expansion in powers of $p$ we have in close analogy with (21) and (22):

$$\begin{aligned} g &= g^{(0)} + pg^{(1)} + p^2 g^{(2)} + \ldots \\ g^{(0)} &= -\int \mathrm{d}z\, h(z)\left[1 - \exp\left\{-\frac{\mathrm{i}z^2}{2}\phi^{+}(E\hat{L} + \mathrm{i}\varepsilon)^{-1}\phi\right\}\right] \\ g^{(1)} &= \int \mathrm{d}y\, h(y) \int \mathrm{d}z\, h(z)\left[-\exp\left(-\frac{\mathrm{i}}{2}z^2\phi^{+}(E\hat{L} + \mathrm{i}\varepsilon)^{-1}\phi\right)\right. \\ &\quad \left. + \exp\left(\frac{\mathrm{i}}{2}z^2\phi^{+}\left[E\hat{L} + \mathrm{i}\varepsilon - \frac{y^2}{E\hat{L} + \mathrm{i}\varepsilon}\right]^{-1}\phi\right)\right]. \end{aligned} \tag{45}$$

Considering, for the sake of convenience, the distribution $h(z)$ to be Gaussian

$$h(z) = \sqrt{\frac{\alpha}{2\pi}}\exp\left(-\frac{\alpha x^2}{2}\right)$$

we get the following expression for $g^{(0)}$:

$$g^{(0)} = \sqrt{\frac{\alpha}{\alpha + \mathrm{i}\phi^{+}\hat{L}(E + \mathrm{i}\varepsilon\hat{L})^{-1}\phi}} - 1. \tag{46}$$

We can see that apart from its singular point ($E = 0$ and $\phi^{+}\hat{L}\phi = 0$) the function $g^{(0)}(\phi)$ depends on the invariant $y = \phi^{+}\hat{L}\phi$ only and is independent of $x = \phi^{+}\phi$. That is why it proves to be invariant with respect to rotations generated by the matrices $\hat{T}$ satisfying condition (32). It is easy to make sure that such an invariance holds for higher order terms in the expansion (45) as well. So there is a single (invariant) saddle-point only contributing to $\langle Z(E, r, J)\rangle$ that leads to the vanishing of the connected part of the level correlator $S_{\rm con}(E, r)$.

Meanwhile, expanding the function $g^{(0)}(\phi)$ to second order terms in $\phi$ we get

$$g^{(0)}(\phi)=\frac{\mathrm{i}}{2\alpha}\,\phi^+\hat{L}(E+\mathrm{i}\varepsilon\hat{L})^{-1}\phi+\mathrm{O}(\phi^4). \tag{47}$$

We see that now $g^{(0)}(\phi)$ depends on both invariants $x$ and $y$ this leads according to (42) to the correct expression for $\rho^{(0)}(E)=\delta(E)$. Thus, in spite of the invalidity of the expansion (47) for $E\to 0$, this way results in a valid expression for the DOS.

The situation is quite analogous for the higher order terms in $p$ as well. The function $g(\phi)$ is singular for $\phi^+\hat{L}\phi=0$ and energy $E$ lying in a spectrum of the matrix $\hat{H}$. When calculating the DOS, we can consider the energy $E$ to have an imaginary part and expand $g(\phi)$ as a power series in $\phi$ along lines similar to (47). If we now let $E$ tend to the real axis, the coefficient of $\phi^+\phi$ in that expansion gives the correct value of $\rho(E)$.

When calculating the correlation function $K(E, r)$ we have, in contrast, to consider the energy $E$ to be a real number from the very beginning, which invalidates an expansion analogous to (47). As we have seen previously, the true function $g(\phi)$ does not depend on $\phi^+\phi$, which means the level correlations vanish.

To analyse the situation at $p>1$ we use the relationship between the sparse RM model and the Anderson model on the Bethe lattice, characterized by the Hamiltonian:

$$\tilde{H}=\sum_i \varepsilon_i c_i^+ c_i+\sum_{\langle ij\rangle} t_{ij} c_i^+ c_j \qquad t_{ij}=t_{ji}^*. \tag{48}$$

For the sake of simplicity we take the diagonal matrix elements of the Hamiltonian of the Anderson model to be equal to zero. As to the nearest-neighbour matrix elements $t_{ij}$, we would consider them to be real identically distributed random numbers with a distribution function $h(t)$. Moreover, let us now consider the branching number of the lattice at any site to be a random variable independently distributed according to the Poisson law with a mean value equal to $p$. It is clear that any non-trivial distribution of diagonal elements $\varepsilon_i$ could be easily included within the scope of the supersymmetric approach in this as well as the RM model. Using this approach we introduce an eight-component supervector $\phi_i$ at every lattice site $i$ and define the partition function analogously to (28). Let $G_l(\phi)$ denote the result of integrating out all variables $\phi$ attached to sites belonging to $l$th branch of the lattice originating in a site $i$ ($l=1,\ldots,k_i$; $k_i$-branching number of the lattice in the site $i$). Then function $G(\phi_j)$, site $j$ being the neighbour of site $i$ towards the origin of the Bethe lattice, is given in terms of $G_l(\phi_i)$ by the following expression:

$$G(\phi_j)=\int[\mathrm{d}\phi_i]\exp\left\{\frac{\mathrm{i}}{2}E\phi_i^+\hat{L}\phi_i-(\varepsilon/2)\phi_i^+\phi_i\right\}\exp[\mathrm{i}t_{ij}\phi_j^+\hat{L}\phi_i]\prod_{l=1}^{ki}G_l(\phi_i). \tag{49}$$

Averaging this equation over disorder and assuming that, after averaging, all the sites of the Bethe lattice are equivalent, we easily get the self-consistent equation for $G_a(\phi)=\langle G(\phi)\rangle$:

$$G_a(\psi)=\int[\mathrm{d}\phi]\tilde{h}(\phi^+\hat{L}\psi)\exp\left\{\frac{\mathrm{i}}{2}E\phi^+\hat{L}\phi-(\varepsilon/2)\phi^+\phi+p[G_a(\phi)-1]\right\}. \tag{50}$$

It is easy to make sure this equation coincides with (31) if we denote $g(\phi)=G_a(\phi)-1$ and take into account that the denominator in (31) is equal to unity.

So, the sparse RM model is intimately related to the Anderson model on the Bethe lattice with a random Poisson-like local branching number. The local topological equivalence between a randomly branching tree and a structure of infinite cluster of sparse RM model was pointed out in [16]. The reason for such an equivalence is that every closed loop in such an infinite cluster includes of order of $\log N$ sites with the probability one. So, the existence of the loops could be neglected when $N \to \infty$.

As is well known [25], there is the localization transition in the Anderson model on the regular Bethe lattice (see also [24] and [26], where such a transition was studied in the frame of the supersymmetric nonlinear $\sigma$-model, which differs from the original Anderson model in view of the DOS being constant independent of the energy). In our approach it manifests itself in the following way [30]. If we consider the regular Bethe lattice with branching number $k$, the self-consistent equation would be analogous to (50) with $G^k(\phi)$ substituted for $\exp p[G_a(\phi)-1]$ on the right-hand side. Within the localized phase this equation at $\varepsilon \to 0$ has a single invariant solution $G(\phi)$ depending only on $\phi^+\hat{L}\phi$. At the transition point spontaneous breaking of the invariance occurs and a set of non-invariant solutions appears, which leads to the Goldstone (diffusion) form of the density–density correlator (cf [24, 26]).

The close similarity of the two models suggests that an Anderson transition also exists in the sparse RM model. Using a series expansion in powers of $1/p$ it can be easily shown that at $p \gg 1$ (31) has a set of non-invariant solutions $g_T(\phi)$, that are non-singular at $\phi = 0$. To the leading order in $1/p$ it has the form:

$$g_T^{(0)}(\phi) = -\mathrm{i}\phi^+\hat{L}^{1/2}\hat{T}^{-1}\hat{\sigma}_0\hat{T}\hat{L}^{1/2}\phi$$
$$\sigma_0 = \frac{1}{4p}(E - \mathrm{i}\hat{L}\sqrt{4pB - E^2}). \tag{51}$$

This manifold of saddle-points has the form described by (33). We conclude, that the localization transition in the sparse RM model occurs with a decrease in the connectivity $p$ at some value $p_l > 1$ (energy $E$ and distribution $h(z)$ being fixed).

The main difference from the Bethe lattice is the existence of finite clusters. As a result, at $p > 1$ the function $g(\phi)$ is a sum of the finite cluster contribution $g_{\mathrm{fin}}(\phi)$ (given by a series in powers of $p$, (45)) and the infinite cluster contribution $g_{\mathrm{inf}}(\phi)$. At $1 < p < p_l$ both $g_{\mathrm{fin}}(\phi)$ and $g_{\mathrm{inf}}(\phi)$ are invariant, i.e. depend on $\phi^+\hat{L}\phi$ only. As it is easy to show, this leads to the following expression for the correlator:

$$K_{\mathrm{con}}(E, r) = \frac{2\pi\mathrm{i}\rho(E)}{r} \tag{52}$$

resulting in the vanishing of $S_{\mathrm{con}}(E, r)$.

At $p > p_l$ the function $g_{\mathrm{inf}}(\phi)$ ceased to be invariant, which corresponds to the delocalized states emerging on the infinite cluster. We get correspondingly

$$K_{\mathrm{con}}(E, r) = \frac{2\pi\mathrm{i}\rho_{\mathrm{fin}}(E)}{r} + K^D_{\mathrm{con}}[\rho_{\mathrm{inf}}(E), r] \tag{53}$$

where the second term is the Dyson correlation function with the full DOS $\rho(E)$ replaced by the contribution to it from the infinite cluster, which could be found by inserting $g_{\mathrm{inf}}(\phi)$ into (42).

Qualitative arguments in favour of the vanishing of level correlations in localized phase were presented in [13]. This effect was recently observed by computer simulation of the two-dimensional disordered tight-binding model with spin–orbit coupling [27].

The question about energy-level correlation in disordered samples was also addressed in [28].

## 4. Conclusion

To study the statistical properties of the sparse RM model we used the supersymmetric approach and observed close similarity between the model under consideration and the Anderson model on the Bethe lattice: in both models there is a delocalization transition manifesting itself as the spontaneous breaking of UOSP(2, 2/2, 2) global invariance by the solution of the basic integral equation.

Depending on the value of the 'mean connectivity' parameter $p$ the behaviour of the system changes crucially. At $p<1$ there are only finite clusters. At $1<p<p_l$, $p_l$ being the delocalization transition point, the infinite cluster also exists, but all the eigenvectors of the corresponding random matrices are localized, which results in the absence of eigenvalue correlations. At $p>p_l$ delocalized eigenstates emerge and the connected correlator of level densities $S_{\mathrm{con}}(E, r)$ acquires the Dyson form with the full DOS replaced by the contribution from the infinite cluster.

It would be interesting to study in more detail the properties of the sparse RM model in the vicinity of the transition point $p_l$ within the scope of the supersymmetric approach as well as by direct computer simulation. We also hope that the supersymmetric approach could be useful for investigating the statistical properties of other classes of RM, e.g. asymmetric random matrices [29], which are relevant to neural networks and quantum chaos problems. We consider all this to be the prospect of future investigations.


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*Note added in proof.* As was pointed out to us by Professor M R Zirnbauer, if the functional integral in (10) is to be well defined, then it should be restricted to all even functions that vanish at the origin. This restriction does not invalidate our derivation since the saddle-point solution $g(\phi)$ does possess both of these properties.

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